- Effective population size:

Let p' = allele frequency after mating

$$Var(p') = \frac{p(1-p)}{2N_e}$$] will demonstrate

N_e = effective population size ← Most results work when you use N_e instead of N

- Wright-Fisher model:

X_i = # A_1 alleles

$X_i \sim Bin(2N, p)$

$p' = \frac{1}{2N} X_i$

$$Var(p') = \frac{1}{(2N)^2} Var(X_i) = \frac{1}{(2N)^2} 2N\, p(1-p) = \frac{p(1-p)}{2N}$$

So in Wright-Fisher $N = N_e$

- Wright-Fisher w/ varying population size

$$N = \begin{cases} N_1 & \text{w.p. } r \\ N_2 & \text{w.p. } 1-r \end{cases}$$

$X_i \mid N_1 \sim Bin(N_1, p)$
$X_i \mid N_2 \sim Bin(N_2, p)$

$$\mathrm{var}\left(\frac{1}{2N}X_i\right) = \underbrace{E\left[\mathrm{var}\left(\frac{X_i}{2N}\,\middle|\,N\right)\right]} + \underbrace{\mathrm{var}\left[\underbrace{E\left(\frac{X_i}{2N}\,\middle|\,N\right)}_{=\frac{p\,2N}{2N} = p}\right]}_{=0}$$

$$\mathrm{var}\left(\frac{X_i}{2N_1}\right) r + \mathrm{var}\left(\frac{X_i}{2N_2}\right)(1-r)$$

$$= \frac{p(1-p)}{2N_1} r + \frac{p(1-p)}{2N_2}(1-r)$$

$$= \frac{p(1-p)}{2\left(r\frac{1}{N_1} + (1-r)\frac{1}{N_2}\right)^{-1}}$$

$$\Rightarrow \boxed{N_e = \left(r\frac{1}{N_1} + (1-r)\frac{1}{N_2}\right)^{-1} = \text{harmonic mean}}$$

↳ harmonic mean is <u>much</u> less than arithmetic mean

↳ so bottlenecks really reduce N_e

§2.8: More Realistic Models:

WTS, that $var(p') = \frac{pq}{2N_e}$ and $\Delta H = \frac{-H}{2N_e}$.

under general conditions,

- Biallelic: A_1, A_2
  allele freq: $p, 1-p$

Let X_i = # offspring for A_1 allele i
Y_j = # offspring for A_2 allele j] may have different distr

X = # A_1 alleles in next generation
Y = # A_2 alleles in next generation

$$X = X_1 + X_2 + \dots + X_{2Np}$$

$$Y = Y_1 + Y_2 + \dots + Y_{2N(1-p)}$$

$$p' = \frac{X}{X+Y}$$

- In Wright-Fisher, X_i and Y_j have same distribution for all i and j.
- In Wright-Fisher, $X+Y$ is fixed. Here it is not.

- First, assume ① $E[X_i] = E[Y_i] = 1$
  ② $var(X_i) = var(Y_i) = \sigma^2$
  ③ all are uncorrelated.

$var(X) = 2Np\sigma^2$
$var(Y) = 2N(1-p)\sigma^2$
$E[X] = 2Np$
$E[Y] = 2N(1-p)$

<u>Goal:</u> $var(p')$

↑ variance of a ratio is tough, we will use approximations

⋮ algebra

$var(p') \approx \frac{p(1-p)\sigma^2}{2N}$ for large N

$\Rightarrow N_e = N/\sigma^2$ where σ^2 is variance of offspring #

- Under this model, can show

$$\Delta H = -\frac{1}{2N_e} H$$

Proof $H = 2p(1-p)$

$\Delta H = 2p'(1-p') - H$

$E(\Delta H) = 2E[p'(1-p')] - H$

$$\text{var}(p') = E(p'^2) + E(p')^2$$

$$E[p'(1-p')] = E[p'] - E[p'^2]$$

$$= E[p'] - \text{var}(p') + E(p')^2$$

$$E(p') \approx p$$

$$\text{var}(p') \approx \frac{p(1-p)}{2N_e}$$

$$= p + p^2 - \frac{p(1-p)}{2Ne}$$

$$= p(1-p) - \frac{p(1-p)}{2N_e}$$

$$2E[p'(1-p')] - H$$

$$= 2p(1-p) - \frac{2p(1-p)}{2N_e} - H$$

$$= H - \frac{H}{2N_e} - H$$

$$= \frac{-H}{2N_e} \quad //$$