Conditional Probability and Independence

David Gerard

2017-09-28

Learning Objectives

- Conditional Probability
- Independence
- Sections 2.1.6 and 2.2 in DBC

Conditional Probability

Probability gives chances for events in outcome set S.

Often: Have partial information about event of interest.

Example: Number of Deaths in the U.S. in 1996

Cause	All ages	1-4	5-14	15-24	25-44	45-64	≥ 65
Heart	733,125	207	341	920	16,261	102,510	612,886
Cancer	544,161	440	1,035	1,642	22,147	132,805	386,092
HIV	32,003	149	174	420	22,795	8,443	22
Accidents[1]	92,998	2,155	3,521	13,872	26,554	16,332	30,564
Homicide[2]	24,486	395	513	6,548	9,261	7,717	52
All causes	2,171,935	5,947	8,465	32,699	148,904	380,396	1,717,218

[1] Accidents and adverse effects, [2] Homicide and legal intervention

Probabilities and conditional probabilities for causes of death:

- $P(\text{accident}) =$
- $P(5 \leq \text{age} \leq 14) =$
- $P(\text{accident and } 5 \leq \text{age} \leq 14) =$
- $P(\text{accident} \mid 5 \leq \text{age} \leq 14) =$

Conditional Probability

Probability gives chances for events in outcome set S.

Often: Have partial information about event of interest.

Example: Number of Deaths in the U.S. in 1996

Cause	All ages	1-4	5-14	15-24	25-44	45-64	$\geq$ 65
Heart	733,125	207	341	920	16,261	102,510	612,886
Cancer	544,161	440	1,035	1,642	22,147	132,805	386,092
HIV	32,003	149	174	420	22,795	8,443	22
Accidents[1]	92,998	2,155	3,521	13,872	26,554	16,332	30,564
Homicide[2]	24,486	395	513	6,548	9,261	7,717	52
All causes	2,171,935	5,947	8,465	32,699	148,904	380,396	1,717,218

[1] Accidents and adverse effects, [2] Homicide and legal intervention

Probabilities and conditional probabilities for causes of death:

- $P(\text{accident}) = 92,998/2,171,935 = 0.04282$
- $P(5 \leq \text{age} \leq 14) =$
- $P(\text{accident and } 5 \leq \text{age} \leq 14) =$
- $P(\text{accident} \mid 5 \leq \text{age} \leq 14) =$

Conditional Probability

Probability gives chances for events in outcome set S.

Often: Have partial information about event of interest.

Example: Number of Deaths in the U.S. in 1996

Cause	All ages	1-4	5-14	15-24	25-44	45-64	$\geq$ 65
Heart	733,125	207	341	920	16,261	102,510	612,886
Cancer	544,161	440	1,035	1,642	22,147	132,805	386,092
HIV	32,003	149	174	420	22,795	8,443	22
Accidents[1]	92,998	2,155	3,521	13,872	26,554	16,332	30,564
Homicide[2]	24,486	395	513	6,548	9,261	7,717	52
All causes	2,171,935	5,947	8,465	32,699	148,904	380,396	1,717,218

[1] Accidents and adverse effects, [2] Homicide and legal intervention

Probabilities and conditional probabilities for causes of death:

- $P(\text{accident}) = 92,998/2,171,935 = 0.04282$
- $P(5 \leq \text{age} \leq 14) = 8,465/2,171,935 = 0.00390$
- $P(\text{accident and } 5 \leq \text{age} \leq 14) =$
- $P(\text{accident} \mid 5 \leq \text{age} \leq 14) =$

Conditional Probability

Probability gives chances for events in outcome set S.

Often: Have partial information about event of interest.

Example: Number of Deaths in the U.S. in 1996

Cause	All ages	1-4	5-14	15-24	25-44	45-64	$\geq$ 65
Heart	733,125	207	341	920	16,261	102,510	612,886
Cancer	544,161	440	1,035	1,642	22,147	132,805	386,092
HIV	32,003	149	174	420	22,795	8,443	22
Accidents[1]	92,998	2,155	3,521	13,872	26,554	16,332	30,564
Homicide[2]	24,486	395	513	6,548	9,261	7,717	52
All causes	2,171,935	5,947	8,465	32,699	148,904	380,396	1,717,218

[1] Accidents and adverse effects, [2] Homicide and legal intervention

Probabilities and conditional probabilities for causes of death:

- $P(\text{accident}) = 92,998/2,171,935 = 0.04282$
- $P(5 \leq \text{age} \leq 14) = 8,465/2,171,935 = 0.00390$
- $P(\text{accident and } 5 \leq \text{age} \leq 14) = 3,521/2,171,935 = 0.00162$
- $P(\text{accident} \mid 5 \leq \text{age} \leq 14) =$

Conditional Probability

Probability gives chances for events in outcome set S.

Often: Have partial information about event of interest.

Example: Number of Deaths in the U.S. in 1996

Cause	All ages	1-4	5-14	15-24	25-44	45-64	$\geq$ 65
Heart	733,125	207	341	920	16,261	102,510	612,886
Cancer	544,161	440	1,035	1,642	22,147	132,805	386,092
HIV	32,003	149	174	420	22,795	8,443	22
Accidents[1]	92,998	2,155	3,521	13,872	26,554	16,332	30,564
Homicide[2]	24,486	395	513	6,548	9,261	7,717	52
All causes	2,171,935	5,947	8,465	32,699	148,904	380,396	1,717,218

[1] Accidents and adverse effects, [2] Homicide and legal intervention

Probabilities and conditional probabilities for causes of death:

- $P(\text{accident}) = 92,998/2,171,935 = 0.04282$
- $P(5 \leq \text{age} \leq 14) = 8,465/2,171,935 = 0.00390$
- $P(\text{accident and } 5 \leq \text{age} \leq 14) = 3,521/2,171,935 = 0.00162$
- $P(\text{accident} \mid 5 \leq \text{age} \leq 14) = 3,521/8,465 = 0.41595$

Conditional Probability

$P(\text{accident} | 5 \leq \text{age} \leq 14)$

Conditional Probability

Conditional probability of A given B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad \text{if } P(B) > 0$$

Conditional Probability

$$P(\text{accident}|5 \leq \text{age} \leq 14) = \frac{3,521}{8,465}$$

Conditional Probability

Conditional probability of A given B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad \text{if } P(B) > 0$$

Conditional Probability

$$P(\text{accident}|5 \leq \text{age} \leq 14) = \frac{3,521}{8,465} = \frac{3,521/2,171,935}{8,465/2,171,935}$$

Conditional Probability

Conditional probability of A given B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad \text{if } P(B) > 0$$

Conditional Probability

$$P(\text{accident}|5 \le \text{age} \le 14) = \frac{3,521}{8,465} = \frac{3,521/2,171,935}{8,465/2,171,935}$$

$$= \frac{P(\text{accident and } 5 \le \text{age} \le 14)}{P(5 \le \text{age} \le 14)}$$

Conditional Probability

Conditional probability of A given B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad \text{if } P(B) > 0$$

Conditional Probability

$\rightsquigarrow$ measure conditional probability with respect to a subset of S

Conditional probability of A given B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad \text{if } P(B) > 0$$

If $P(B) = 0$ then $P(A|B)$ is undefined.

$\rightsquigarrow$

Multiplication rule:

$P(A \cap B) = P(A \mid B) \times P(B)$.

Independence

Example: Roll two fair dice

What is probability that 2nd die shows ⚀? $P(\text{2nd die} = ⚀) = \frac{1}{6}$

What is probability 2nd die shows ⚀ if 1st die showed ⚀?

$$P(\text{2nd die} = ⚀ \mid \text{1st die} = ⚀) = \frac{1}{6}$$

...and if the 1st die did not show ⚀?

$$P(\text{2nd die} = ⚀ \mid \text{1st die} \neq ⚀) = \frac{1}{6}$$

Chance of ⚀ on 2nd die the same, no matter what the 1st.

Independence

The event A is **independent** of the event B if its chances are not affected by the occurrence of B,

$$P(A|B) = P(A).$$

You can show that the following definitions of independence are equivalent.

Independence

Events A and B are independent if

$$P(A|B) = P(A) \text{ if and only if}$$
$$P(B|A) = P(B) \text{ if and only if}$$
$$P(A \cap B) = P(A) \times P(B)$$

Independence

Suppose event A is independent of event B.

Then, knowing that B has occurred does not effect the probability of event A occurring: $P(A|B) = P(A)$. Now,

$$P(B|A)$$

Thus, event B is independent of event A.

The argument in the other direction is exactly the same.
So, the following two statements are equivalent:

$$P(A|B) = P(A) \qquad \text{and} \qquad P(B|A) = P(B)$$

and we simply state that events A and B are independent.

Independence

Suppose event A is independent of event B.

Then, knowing that B has occurred does not effect the probability of event A occurring: $P(A|B) = P(A)$. Now,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Thus, event B is independent of event A.

The argument in the other direction is exactly the same.
So, the following two statements are equivalent:

$$P(A|B) = P(A) \qquad \text{and} \qquad P(B|A) = P(B)$$

and we simply state that events A and B are independent.

Independence

Suppose event A is independent of event B.

Then, knowing that B has occurred does not effect the probability of event A occurring: $P(A|B) = P(A)$. Now,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

Thus, event B is independent of event A.

The argument in the other direction is exactly the same.
So, the following two statements are equivalent:

$$P(A|B) = P(A) \qquad \text{and} \qquad P(B|A) = P(B)$$

and we simply state that events A and B are independent.

Independence

Suppose event A is independent of event B.

Then, knowing that B has occurred does not effect the probability of event A occurring: $P(A|B) = P(A)$. Now,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)} = \frac{P(A)P(B)}{P(A)}$$

Thus, event B is independent of event A.

The argument in the other direction is exactly the same.
So, the following two statements are equivalent:

$$P(A|B) = P(A) \qquad \text{and} \qquad P(B|A) = P(B)$$

and we simply state that events A and B are independent.

Independence

Suppose event A is independent of event B.

Then, knowing that B has occurred does not effect the probability of event A occurring: $P(A|B) = P(A)$. Now,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

Thus, event B is independent of event A.

The argument in the other direction is exactly the same.
So, the following two statements are equivalent:

$$P(A|B) = P(A) \qquad \text{and} \qquad P(B|A) = P(B)$$

and we simply state that events A and B are independent.

Independence

Also, if A and B are independent events, then

$$P(A \cap B)$$

and in the other direction... If $P(A \cap B) = P(A)P(B)$, then

$$P(A|B)$$

Thus, the following three statements are equivalent definitions of independence of events A and B :

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A) \times P(B)$$

Independence

Also, if A and B are independent events, then

$$P(A \cap B) = P(A|B)P(B)$$

and in the other direction... If $P(A \cap B) = P(A)P(B)$, then

$$P(A|B)$$

Thus, the following three statements are equivalent definitions of independence of events A and B :

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A) \times P(B)$$

Independence

Also, if A and B are independent events, then

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

and in the other direction... If $P(A \cap B) = P(A)P(B)$, then

$$P(A|B)$$

Thus, the following three statements are equivalent definitions of independence of events A and B :

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A) \times P(B)$$

Independence

Also, if A and B are independent events, then

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

and in the other direction... If $P(A \cap B) = P(A)P(B)$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Thus, the following three statements are equivalent definitions of independence of events A and B :

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A) \times P(B)$$

Independence

Also, if A and B are independent events, then

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

and in the other direction... If $P(A \cap B) = P(A)P(B)$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)}$$

Thus, the following three statements are equivalent definitions of independence of events A and B :

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A) \times P(B)$$

Independence

Also, if A and B are independent events, then

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

and in the other direction... If $P(A \cap B) = P(A)P(B)$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

Thus, the following three statements are equivalent definitions of independence of events A and B :

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A) \times P(B)$$

Comparison: Independence vs Disjoint Events

- Disjoint event are not independent: for disjoint events, even if $P(A) > 0$, $P(A|B) = 0$.
- Disjoint events can use a special case of the inclusion/exclusion formula: $P(A \cup B) = P(A) + P(B) - P(AandB) = P(A) + P(B)$.
- I sometimes call this the "or" rule.
- Independent events can use a special case of the multiplication rule: $P(A \cap B) = P(A|B)P(B) = P(A)P(B)$.
- I sometimes call this the "and" rule.
- Note: Independence cannot be displayed as a Venn diagram because it depends not just on the outcomes that make up the events, but also the probabilities of the events.

Exercise

If A and B are two independent events, prove that A^c is independent of B^c.