Conditional Probability and Independence

David Gerard 2017-09-28

- Conditional Probability
- Independence
- Sections 2.1.6 and 2.2 in DBC

Probability gives chances for events in outcome set S.

Often: Have partial information about event of interest.

Example: Number of Deaths in the U.S. in 1996

Cause	All ages	1-4	5-14	15-24	25-44	45-64	\geq 65
Heart	733,125	207	341	920	16,261	102,510	612,886
Cancer	544,161	440	1,035	1,642	22,147	132,805	386,092
HIV	32,003	149	174	420	22,795	8,443	22
Accidents ¹	92,998	2,155	3,521	13,872	26,554	16,332	30,564
Homicide ²	24,486	395	513	6,548	9,261	7,717	52
All causes	2,171,935	5,947	8,465	32,699	148,904	380,396	1,717,218

 1 Accidents and adverse effects, 2 Homicide and legal intervention

Probabilities and conditional probabilities for causes of death:

- P(accident) =
- $P(5 \le age \le 14) =$
- $P(\text{accident and } 5 \leq \text{age} \leq 14) =$
- $P(\text{accident} \mid 5 \leq \text{age} \leq 14) =$

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Probabilities and conditional probabilities for causes of death:

- P(accident) = 92,998/2,171,935 = 0.04282
- $P(5 \le age \le 14) =$
- P(accident and $5 \le$ age $\le 14) =$
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Probabilities and conditional probabilities for causes of death:

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- $P(\text{accident} \mid 5 \le \text{age} \le 14) = 3,521/8,465 = 0.41595$

 $P(\text{accident}|5 \leq \text{age} \leq 14)$

Conditional Probability

$$P(A \mid B) = rac{P(A \cap B)}{P(B)}, \qquad ext{if } P(B) > 0$$

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$$\begin{aligned} P(\text{accident}|5 \le \text{age} \le 14) &= \frac{3,521}{8,465} = \frac{3,521/2,171,935}{8,465/2,171,935} \\ &= \frac{P(\text{accident and } 5 \le \text{age} \le 14)}{P(5 \le \text{age} \le 14)} \end{aligned}$$

$$P(A \mid B) = rac{P(A \cap B)}{P(B)}, \qquad ext{if } P(B) > 0$$

\rightsquigarrow measure conditional probability with respect to a subset of S

Conditional probability of A given B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{if } P(B) > 0$$

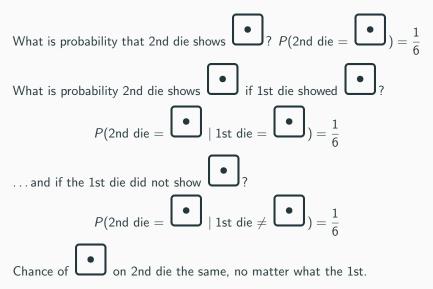
If P(B) = 0 then P(A|B) is undefined.

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Multiplication rule:

 $P(A \cap B) = P(A \mid B) \times P(B).$

Example: Roll two fair dice



The event A is **independent** of the event B if its chances are not affected by the occurrence of B,

P(A|B) = P(A).

You can show that the following definitions of independence are equivalent.

Independence

Events A and B are independent if

P(A|B) = P(A) if and only if P(B|A) = P(B) if and only if $P(A \cap B) = P(A) \times P(B)$

Suppose event A is independent of event B.

Then, knowing that B has occurred does not effect the probability of event A occurring: P(A|B) = P(A). Now,

P(B|A)

Thus, event B is independent of event A.

The argument in the other direction is exactly the same. So, the following two statements are equivalent:

P(A|B) = P(A) and P(B|A) = P(B)

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The argument in the other direction is exactly the same. So, the following two statements are equivalent:

$$P(A|B) = P(A)$$
 and $P(B|A) = P(B)$

Also, if A and B are independent events, then $P(A \cap B)$

and in the other direction... If $P(A \cap B) = P(A)P(B)$, then

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Thus, the following three statements are equivalent definitions of independence of events A and B:

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Comparison: Independence vs Disjoint Events

- Disjoint event are not independent: for disjoint events, even if P(A) > 0, P(A|B) = 0.
- Disjoint events can use a special case of the inclusion/exclusion formula:
 P(A ∪ B) = P(A) + P(B) P(AandB) = P(A) + P(B).
- I sometimes call this the "or" rule.
- Independent events can use a special case of the multiplication rule: P(A ∩ B) = P(A|B)P(B) = P(A)P(B).
- I sometimes call this the "and" rule.
- Note: Independence cannot be displayed as a Venn diagram because it depends not just on the outcomes that make up the events, but also the probabilities of the events.

If A and B are two independent events, prove that A^c is independent of B^c .