Bayes Theorem

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Learning Objectives

- Bayes Rule
- Section 2.2.7 of DBC

Motivation

- There are many times that we want $P(B|A)$.
- However, we might only have information on $P(A|B)$.
- E.g. from medical tests, we often have a lot of knowledge of the probability of a test resulting positive given an patient has a disease, or the probability of a test resulting positive given a patient does not have a disease.
- But when a test is run, we want the probability that a patient has a disease given that a test is positive or negative.

Motivation

Data from OpenIntro p98. Breast cancer for women in Canada.

$$P(\text{positive}|\text{cancer}) = 0.89 \text{ so } P(\text{negative}|\text{cancer}) = 0.11.$$
$$P(\text{positive}|\text{not cancer}) = 0.07 \text{ so } P(\text{negative}|\text{not cancer}) = 0.93.$$
$$P(\text{cancer}) = 0.0035 \text{ so } P(\text{not cancer}) = 0.9965$$

- But we want to know $P(\text{cancer}|\text{positive})$.

Bayes Rule

Recall Multiplication Rule:

$$P(B|A)P(A) = P(A \cap B)$$

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$$P(B|A)P(A) = P(A \cap B) = P(A|B)P(B)$$
$$P(B|A)P(A) = P(A|B)P(B)$$

Bayes Rule

Recall Multiplication Rule:

$$
\begin{aligned}
P(B|A)P(A) &= P(A \cap B) = P(A|B)P(B) \\
P(B|A)P(A) &= P(A|B)P(B) \\
P(B|A) &= \frac{P(A|B)P(B)}{P(A)}
\end{aligned}
$$

Bayes Rule

Recall Multiplication Rule:

$$P(B|A)P(A) = P(A \cap B) = P(A|B)P(B)$$
$$P(B|A)P(A) = P(A|B)P(B)$$
$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

This is known as Bayes rule.

Bayes Rule

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Law of Total Probability

You almost always get $P(A)$ by using the law of total probablity:

Law of total probability (more general form)

Suppose $B_1, B_2, \ldots, B_K$ is a partition of the sample space S. I.e. $B_1 \cup B_2 \cup \cdots \cup B_K = S$ and $B_i \cap B_j = \emptyset$ for all $i \neq j$, then

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_K) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_K)P(B_K). \end{aligned}$$

We previously defined this law using $K = 2$:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

Our Cancer Example

$$P(\text{positive}|\text{cancer}) = 0.89 \text{ so } P(\text{negative}|\text{cancer}) = 0.11.$$
$$P(\text{positive}|\text{not cancer}) = 0.07 \text{ so } P(\text{negative}|\text{not cancer}) = 0.93.$$
$$P(\text{cancer}) = 0.0035 \text{ so } P(\text{not cancer}) = 0.9965$$

$P(\text{cancer}|\text{positive}) = \frac{P(\text{positive}|\text{cancer})P(\text{cancer})}{P(\text{positive})}$.

We need

$$\begin{aligned} P(\text{positive}) &= P(\text{positive}|\text{cancer})P(\text{cancer}) \\ &\quad + P(\text{positive}|\text{not cancer})P(\text{not cancer}) \\ &= 0.89 * 0.0035 + 0.07 * 0.9965 = 0.07287 \end{aligned}$$

Our Cancer Example

$$P(\text{positive}|\text{cancer}) = 0.89 \text{ so } P(\text{negative}|\text{cancer}) = 0.11.$$

$$P(\text{positive}|\text{not cancer}) = 0.07 \text{ so } P(\text{negative}|\text{not cancer}) = 0.93.$$

$$P(\text{cancer}) = 0.0035 \text{ so } P(\text{not cancer}) = 0.9965$$

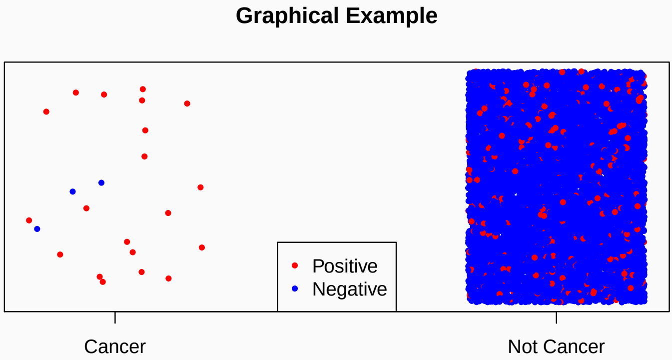
$$P(\text{positive}) = 0.07287$$

$$\begin{aligned} P(\text{cancer}|\text{positive}) &= \frac{P(\text{positive}|\text{cancer})P(\text{cancer})}{P(\text{positive})} \\ &= \frac{0.89 * 0.0035}{0.07287} \\ &= 0.04275 \end{aligned}$$

Intuition

- So the probability you have cancer given a positive test is only about 4%!
- Even though the test is fairly accurate, because there are so many more people who do not have cancer than who have cancer, they make up a majority of the population who have a positive test result.

Graphical Example



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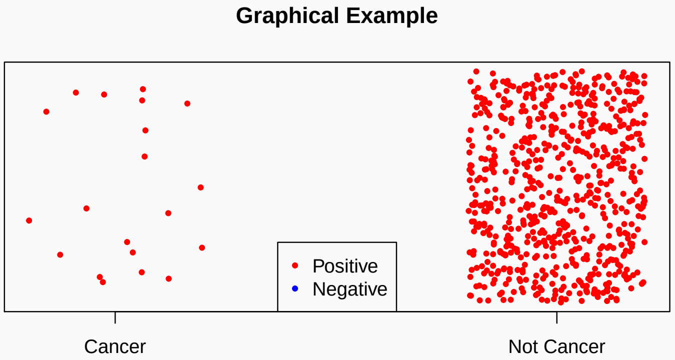
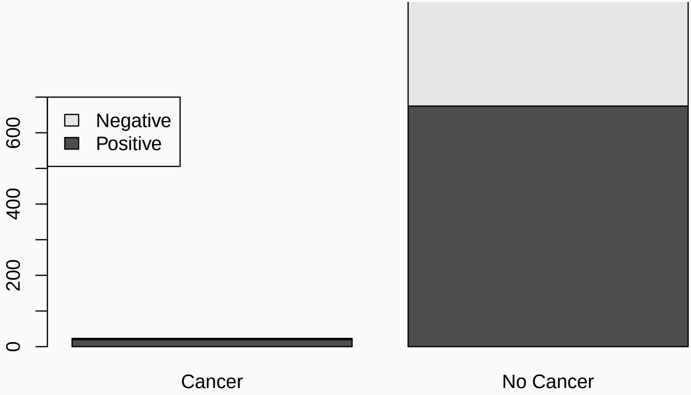


Graphical Example

Zooming In



Another Example

From `email` dataset, we get

$$P(\text{no number}|\text{spam}) = 0.4062$$
$$P(\text{small number}|\text{not spam}) = 0.7482$$
$$P(\text{big number}|\text{not spam}) = 0.1393$$
$$P(\text{spam}) = 0.0936$$

What proportion of emails with no number are spam?

(on chalk board)