Deriving the Binomial PMF

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Most slides borrowed from Linda Collins

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Learning Objectives

- Bernoulli Random Variables
- Binomial Random Variables
- Sections 3.3.1 and 3.4 of DBC

A formula

- Sometimes, if you are lucky, the pmf may be written as an equation in terms of the value of the random variable.
- For the coin flipping example, we will derive this formula.
- Useful beyond just coins: What is the probabiliy of having 3 girls out of 4 children? I.e. many random variables *follow the same distribution*.

Using counting rules to determine probabilities i

Since outcomes in the random flip are equally likely, we just counted the outcomes to determine event probabilities.

Let's generalize the counting process for this probability model.

We want a formula for the number of outcomes having k heads out of 4 flips.

We begin with a discussion of
permutations and
combinations ... also called **binomial coefficients**

Permutations:

How many ways to order a group of 4 people?

4 choices for 1st person $\times$ (3 for 2nd) $\times$ (2 for 3rd) $\times$ (1 for 4th.)

How many ways to order a group of n people?

$$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$$

(Note: $n!$ is pronounced "n factorial.")

Combinations

Combinations, also called **Binomial Coefficients:**

(Let $n = 5$ for the moment just for this one-slide example.)

How many ways to choose a committee of 2 from a group of 5?

$$\frac{\text{5 choices for 1st committee member } \times \text{4 for 2nd}}{\text{2! orderings of 2 person committee}}$$
$$= \frac{5 \cdot 4 \cdot (3 \cdot 2 \cdot 1)}{2!\,(3 \cdot 2 \cdot 1)} = \frac{5!}{2!\,3!}$$

How many ways to choose a committee of k from a group of n?

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Note: $\binom{n}{k}$ is pronounced "n choose k."

Counting outcomes

Counting outcomes for 4 flips of a coin:

How many outcomes with 2 heads out of 4 flips?

Each sequence has four flips: $\{\text{First}, \text{Second}, \text{Third}, \text{Fourth}\}$.

How many outcomes have two heads

That is, now many ways can we choose two locations from four: $\{\text{First}, \text{Second}, \text{Third}, \text{Fourth}\}$?

There are $\binom{n}{k} = \binom{4}{2}$ ways!

Verify

- We can verify this directly
- HHTT, HTHT, HTTH, THHT, THTH, TTHH
- But this formula always works without having to directly count outcomes.

pmf

How many outcomes total?

2 choices for 1st $\times$(2 for 2nd)$\times$(2 for 3rd)$\times$(2 for 4th) $= 2^4 = 16$.

Probability mass function for X:

$$f(x) = \mathbf{P}(X = x) = \frac{\#\text{outcomes w/ } x \text{ heads}}{\#\text{outcomes in total}} = \frac{\binom{4}{x}}{2^4}.$$

Using counting rules to determine probabilities

So, the number of outcomes satisfying $X = 2$ is

$$\frac{4!}{2!\,2!} = \frac{4!}{2!\,(4-2)!} = \binom{4}{2} = \text{"4 choose 2"}$$
$$= 6$$
$$= \text{the number of ways to arrange 2 heads among 4 flips}$$

$$\begin{aligned} S = \{&HHHH, \\ &HHHT, HHTH, HTHH, THHH, \\ &HHTT, HTHT, HTTH, THTH, TTHH, THHT, \\ &HTTT, THTT, TTHT, TTTH, \\ &TTTT\}. \end{aligned}$$

Using counting rules to determine probabilities

The number of ways to arrange x heads among 4 flips is $\binom{4}{x}$.

How many outcomes have 3 Heads?

Using counting rules to determine probabilities

The number of ways to arrange x heads among 4 flips is $\binom{4}{x}$.

How many outcomes have 3 Heads?

$$\binom{4}{3} = \frac{4!}{3!\,(4-3)!} = \frac{24}{6(1)} = 4.$$

Using counting rules to determine probabilities

Note that in real life, it's not quite true that the probability of having a boy $P(M)$ is equal to the probability of having a girl $P(F)$.

If $P(M) \neq P(F)$,

are all 4 outcomes with 3 females equally likely?

$$(FFFM, FFMF, FMFF, MFFF)$$

What is $P(FMFF)$?

Using counting rules to determine probabilities

What is $P(FMFF)$?

$$P(FMFF) = P(MFF \mid F) \times P(F)$$

Using counting rules to determine probabilities

What is $P(FMFF)$?

$$\begin{aligned} P(FMFF) &= P(MFF \mid F) \times P(F) \\ &= P(MFF) \times P(F) \qquad (*) \end{aligned}$$

$(*)$ by independence of gender by birth order

Using counting rules to determine probabilities

What is $P(FMFF)$?

$$
\begin{aligned}
P(FMFF) &= P(MFF \mid F) \times P(F) \\
&= P(MFF) \times P(F) \qquad (*) \\
&= [P(FF \mid M) \times P(M)] \times P(F)
\end{aligned}
$$

$(*)$ by independence of gender by birth order

Using counting rules to determine probabilities

What is $P(FMFF)$?

$$\begin{aligned} P(FMFF) &= P(MFF \mid F) \times P(F) \\ &= P(MFF) \times P(F) && (*) \\ &= [P(FF \mid M) \times P(M)] \times P(F) \\ &= P(FF) \times P(M) \times P(F) && (*) \end{aligned}$$

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Using counting rules to determine probabilities

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Using counting rules to determine probabilities

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&= P(F) \times P(F) \times P(M) \times P(F) && (*)
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Using counting rules to determine probabilities

What is $P(FMFF)$?

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$(*)$ by independence of gender by birth order

Using counting rules to determine probabilities

What is $P(FMFF)$?

$$\begin{aligned}
P(FMFF) &= P(MFF \mid F) \times P(F) \\
&= P(MFF) \times P(F) && (*) \\
&= [P(FF \mid M) \times P(M)] \times P(F) \\
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&= [P(F \mid F) \times P(F)] \times P(M) \times P(F) \\
&= P(F) \times P(F) \times P(M) \times P(F) && (*) \\
&= P(F)^3 \times P(M)
\end{aligned}$$

$(*)$ by independence of gender by birth order

Finally, $P(X = 3) = \binom{4}{3} P(F)^3\, P(M)$

Using counting rules to determine probabilities

So, even if genders are not equally likely,

we can find probabilities for $X = 0, 1, 2, 3$, and 4.

First, let $p = P(F) \qquad (0 < p < 1)$

then $P(M) = 1 - p$,

where $0 \leq p \leq 1$ is the probability of "success" (female birth).

Using counting rules to determine probabilities

$$P(X=3) = \binom{4}{3} P(F)^3 \; P(M)^1$$
$$= \binom{4}{3} p^3 \; (1-p)^{4-3} = 6 \; p^3 \; (1-p)$$

$$P(X=0) =$$

$$P(X=1) =$$

$$P(X=2) =$$

$$P(X=4) =$$

Using counting rules to determine probabilities

$$P(X=3) = \binom{4}{3} P(F)^3 \; P(M)^1$$

$$= \binom{4}{3} p^3 \; (1-p)^{4-3} = 6 \; p^3 \; (1-p)$$

$$P(X=0) = \binom{4}{0} p^0 \; (1-p)^{4-0} = (1-p)^4$$

$$P(X=1) =$$

$$P(X=2) =$$

$$P(X=4) =$$

Using counting rules to determine probabilities

$$P(X = 3) = \binom{4}{3} P(F)^3 \; P(M)^1$$
$$= \binom{4}{3} p^3 \; (1 - p)^{4-3} = 6 \; p^3 \; (1 - p)$$

$$P(X = 0) = \binom{4}{0} p^0 \; (1 - p)^{4-0} = (1 - p)^4$$

$$P(X = 1) = \binom{4}{1} p^1 \; (1 - p)^{4-1} = 4 \; p \; (1 - p)^3$$

$$P(X = 2) =$$

$$P(X = 4) =$$

Using counting rules to determine probabilities

$$P(X=3) = \binom{4}{3}\ P(F)^3\ P(M)^1$$

$$= \binom{4}{3}\ p^3\ (1-p)^{4-3} = 6\ p^3\ (1-p)$$

$$P(X=0) = \binom{4}{0}\ p^0\ (1-p)^{4-0} = (1-p)^4$$

$$P(X=1) = \binom{4}{1}\ p^1\ (1-p)^{4-1} = 4\ p\ (1-p)^3$$

$$P(X=2) = \binom{4}{2}\ p^2\ (1-p)^{4-2} = 4\ p^2\ (1-p)^2$$

$$P(X=4) =$$

Using counting rules to determine probabilities

$$P(X=3) = \binom{4}{3} P(F)^3 \; P(M)^1$$
$$= \binom{4}{3} p^3 \; (1-p)^{4-3} = 6 \; p^3 \; (1-p)$$

$$P(X=0) = \binom{4}{0} p^0 \; (1-p)^{4-0} = (1-p)^4$$

$$P(X=1) = \binom{4}{1} p^1 \; (1-p)^{4-1} = 4 \; p \; (1-p)^3$$

$$P(X=2) = \binom{4}{2} p^2 \; (1-p)^{4-2} = 4 \; p^2 \; (1-p)^2$$

$$P(X=4) = \binom{4}{4} p^4 \; (1-p)^{4-4} = p^4$$

Using counting rules to determine probabilities

Does this agree with our earlier work when $P(F) = p(M) = 0.5$?

Then, $p = 0.5$ and $(1 - p) = 0.5$.

$$
\begin{aligned}
P(X = 0) &= (1 - p)^4 & &= (0.5)^4 = 1/16 \\
P(X = 1) &= 4\, p\, (1 - p)^3 & &= 4\, (0.5)\, (0.5)^3 = 4/16 \\
P(X = 2) &= 4\, p^2\, (1 - p)^2 & &= 6\, (0.5)^2\, (0.5)^2 = 6/16 \\
P(X = 3) &= 6\, p^3\, (1 - p) & &= 4\, (0.5)^3\, (0.5) = 4/16 \\
P(X = 4) &= p^4 & &= (0.5)^4 = 1/16
\end{aligned}
$$

Same probability distribution as before! That's comforting.

Bernoulli and binomial probability distributions

A Bernoulli random variable models a very simple process.
For example, $Y =$ the number of females in one birth. Then,

$$Y = \begin{cases} 1 & \text{if child is female} \\ 0 & \text{if child is male} \end{cases}$$

where $P(Y = 1) = P(F) = p$, and $P(Y = 0) = P(M) = 1 - p$.

We say that $Y \sim \text{Bernoulli}(p)$,
or Y is a Bernoulli random variable with "success" probability p.

Bernoulli and binomial probability distributions

Let $Y = \#$ of "successes" in one Bernoulli (p) "trial"

Then $Y \sim \text{Bernoulli}(p)$ and the pmf for Y is

$$f(y) =$$

Let $X = \#$ of "successes" in n independent Bernoulli (p) "trials"

Then, we say that $X \sim \text{binom}(n, p)$,
or X is a binomial random variable with n **independent** trials and success probability p and the pmf for X is

$$f(x) =$$

Bernoulli and binomial probability distributions

Let $Y = \#$ of "successes" in one Bernoulli (p) "trial"

Then $Y \sim \text{Bernoulli}(p)$ and the pmf for Y is

$$f(y) = p^y\,(1-p)^{1-y} \qquad \text{for } y = 0, 1$$

Let $X = \#$ of "successes" in n independent Bernoulli (p) "trials"

Then, we say that $X \sim \text{binom}(n, p)$,
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Bernoulli and binomial probability distributions

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Let $X = \#$ of "successes" in n independent Bernoulli (p) "trials"

Then, we say that $X \sim \text{binom}(n, p)$,
or X is a binomial random variable with n **independent** trials and success probability p and the pmf for X is

$$f(x) = \binom{n}{x} p^x \,(1-p)^{n-x} \qquad \text{for } x = 0, 1, ..., n$$

The Binomial Expansion

The coefficients in the expansion match those in Pascal's Triangle:

$$(w+y)^1 = 1w^1 + \ 1y^1$$
$$(w+y)^2 = 1w^2 + 2wy + \ \ 1y^2$$
$$(w+y)^3 = 1w^3 + 3w^2y + \ 3w^1y^2 + \ 1y^3$$
$$(w+y)^4 = 1w^4 + 4w^3y + \ 6w^2y^2 + 4wy^3 + 1y^4$$
$$(w+y)^5 = 1w^5 + 5w^4y + 10w^3y^2 + 10w^2y^3 + 5wy^4 + 1y^5$$

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{0} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\ & & & & & \vdots & & & & & \end{array} = \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \vdots & & & & & \end{array}$$

The Binomial Expansion

In general,

$$(w+y)^n = \underbrace{(w+y)(w+y)\dots(w+y)}_{n \text{ factors}} = \sum_{x=0}^{n} \binom{n}{x} w^x y^{n-x}$$

General idea:

$w^5y^3 = wwwwwyyy = wwwwywyy = \cdots = yyywwwww$

The Binomial Expansion

In general,

$$(w+y)^n = \underbrace{(w+y)(w+y)\dots(w+y)}_{n \text{ factors}} = \sum_{x=0}^{n} \binom{n}{x} w^x y^{n-x}$$

General idea:

$w^5y^3 = wwwwwyyy = wwwwywyy = \cdots = yyywwwww$

This result guarantees that the binomial RV has a valid pmf.

To see this, let $w = p, y = (1-p)$. Then, $\displaystyle\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

$$= \sum_{x=0}^{n} \binom{n}{x} w^x y^{n-x} = (w+y)^n = (p + (1-p))^n = 1^n = 1$$

The probabilities for any valid pmf must sum to 1.

Multiple Random Variables, Same Sample Space

We can define several random variables on this same experiment (the same sample space):

$X =$ Number of female children

$Y =$ Number of male children before the first female child is born

$$Z = \begin{cases} 1 & \text{if more female children than male} \\ 0 & \text{otherwise} \end{cases}$$

Outcome	X	Y	Z	Outcome	X	Y	Z	Outcome	X	Y	Z
FFFF	4	0	1	FFMM	2	0	0	FMMM	1	0	0
				FMFM	2	0	0	MFMM	1	1	0
FFFM	3	0	1	FMMF	2	0	0	MMFM	1	2	0
FFMF	3	0	1	MFMF	2	1	0	MMMF	1	3	0
FMFF	3	0	1	MFFM	2	1	0				
MFFF	3	1	1	MMFF	2	2	0	MMMM	0	4	0

Multiple pmfs, Same Sample Space

We can define several random variables and their corresponding pmfs on this same experiment (the same sample space):

Outcome	X	Y	Z	Outcome	X	Y	Z	Outcome	X	Y	Z
FFFF	4	0	1	FFMM	2	0	0	FMMM	1	0	0
				FMFM	2	0	0	MFMM	1	1	0
FFFM	3	0	1	FMMF	2	0	0	MMFM	1	2	0
FFMF	3	0	1	MFMF	2	1	0	MMMF	1	3	0
FMFF	3	0	1	MFFM	2	1	0				
MFFF	3	1	1	MMFF	2	2	0	MMMM	0	4	0

x	0	1	2	3	4
$f(x)$	1/16	4/16	6/16	4/16	1/16

y
$f(y)$

z
$f(z)$

Multiple pmfs, Same Sample Space

We can define several random variables and their corresponding pmfs on this same experiment (the same sample space):

Outcome	X	Y	Z	Outcome	X	Y	Z	Outcome	X	Y	Z
FFFF	4	0	1	FFMM	2	0	0	FMMM	1	0	0
				FMFM	2	0	0	MFMM	1	1	0
FFFM	3	0	1	FMMF	2	0	0	MMFM	1	2	0
FFMF	3	0	1	MFMF	2	1	0	MMMF	1	3	0
FMFF	3	0	1	MFFM	2	1	0				
MFFF	3	1	1	MMFF	2	2	0	MMMM	0	4	0

x	0	1	2	3	4
$f(x)$	1/16	4/16	6/16	4/16	1/16

y	0
$f(y)$	8/16

z	
$f(z)$	

Multiple pmfs, Same Sample Space

We can define several random variables and their corresponding pmfs on this same experiment (the same sample space):

Outcome	X	Y	Z	Outcome	X	Y	Z	Outcome	X	Y	Z
FFFF	4	0	1	FFMM	2	0	0	FMMM	1	0	0
				FMFM	2	0	0	MFMM	1	1	0
FFFM	3	0	1	FMMF	2	0	0	MMFM	1	2	0
FFMF	3	0	1	MFMF	2	1	0	MMMF	1	3	0
FMFF	3	0	1	MFFM	2	1	0				
MFFF	3	1	1	MMFF	2	2	0	MMMM	0	4	0

x	0	1	2	3	4
$f(x)$	1/16	4/16	6/16	4/16	1/16

y	0	1
$f(y)$	8/16	4/16

z
$f(z)$

Multiple pmfs, Same Sample Space

We can define several random variables and their corresponding pmfs on this same experiment (the same sample space):

Outcome	X	Y	Z	Outcome	X	Y	Z	Outcome	X	Y	Z
FFFF	4	0	1	FFMM	2	0	0	FMMM	1	0	0
				FMFM	2	0	0	MFMM	1	1	0
FFFM	3	0	1	FMMF	2	0	0	MMFM	1	2	0
FFMF	3	0	1	MFMF	2	1	0	MMMF	1	3	0
FMFF	3	0	1	MFFM	2	1	0				
MFFF	3	1	1	MMFF	2	2	0	MMMM	0	4	0

x	0	1	2	3	4
$f(x)$	1/16	4/16	6/16	4/16	1/16

y	0	1	2	3	4
$f(y)$	8/16	4/16	2/16	1/16	1/16

z	
$f(z)$	

Multiple pmfs, Same Sample Space

We can define several random variables and their corresponding pmfs on this same experiment (the same sample space):

Outcome	X	Y	Z	Outcome	X	Y	Z	Outcome	X	Y	Z
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				FMFM	2	0	0	MFMM	1	1	0
FFFM	3	0	1	FMMF	2	0	0	MMFM	1	2	0
FFMF	3	0	1	MFMF	2	1	0	MMMF	1	3	0
FMFF	3	0	1	MFFM	2	1	0				
MFFF	3	1	1	MMFF	2	2	0	MMMM	0	4	0

x	0	1	2	3	4
$f(x)$	1/16	4/16	6/16	4/16	1/16

y	0	1	2	3	4
$f(y)$	8/16	4/16	2/16	1/16	1/16

z	0
$f(z)$	11/16

Multiple pmfs, Same Sample Space

We can define several random variables and their corresponding pmfs on this same experiment (the same sample space):

Outcome	X	Y	Z	Outcome	X	Y	Z	Outcome	X	Y	Z
FFFF	4	0	1	FFMM	2	0	0	FMMM	1	0	0
				FMFM	2	0	0	MFMM	1	1	0
FFFM	3	0	1	FMMF	2	0	0	MMFM	1	2	0
FFMF	3	0	1	MFMF	2	1	0	MMMF	1	3	0
FMFF	3	0	1	MFFM	2	1	0				
MFFF	3	1	1	MMFF	2	2	0	MMMM	0	4	0

x	0	1	2	3	4
$f(x)$	1/16	4/16	6/16	4/16	1/16

y	0	1	2	3	4
$f(y)$	8/16	4/16	2/16	1/16	1/16

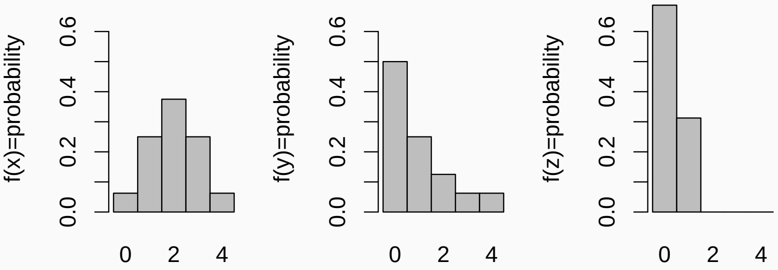
z	0	1
$f(z)$	11/16	5/16

Probability Histograms

x	0	1	2	3	4
$f(x)$	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

y	0	1	2	3	4
$f(y)$	$8/16$	$4/16$	$2/16$	$1/16$	$1/16$

z	0	1
$f(z)$	$11/16$	$5/16$



Recall: Mean

Mean (Expected Value)

Suppose X is a discrete random variable, then

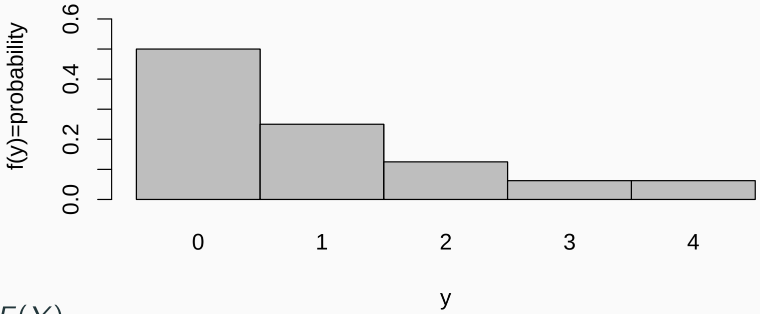
$$\text{mean of } X = E[X] = \sum_{\text{all } x} xP(X = x) = \sum_{\text{all } x} xf(x) = \mu$$

Expected Value (Mean of a r.v.)

Find expected value (mean) of random variable Y.

$Y = \#$ of male children before the first female child is born

y	0	1	2	3	4
$f(y)$	$8/16$	$4/16$	$2/16$	$1/16$	$1/16$



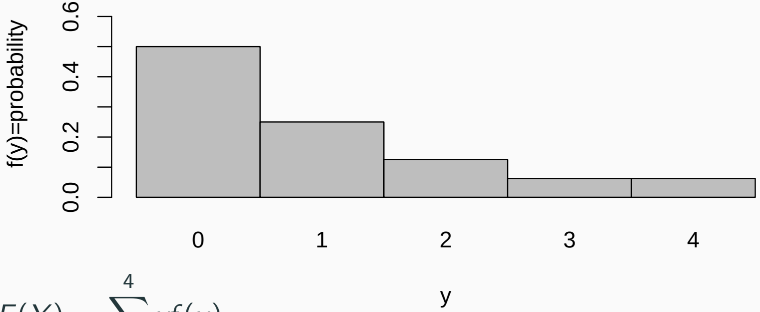
$E(Y) =$

Expected Value (Mean of a r.v.)

Find expected value (mean) of random variable Y.

$Y = \#$ of male children before the first female child is born

y	0	1	2	3	4
$f(y)$	$8/16$	$4/16$	$2/16$	$1/16$	$1/16$



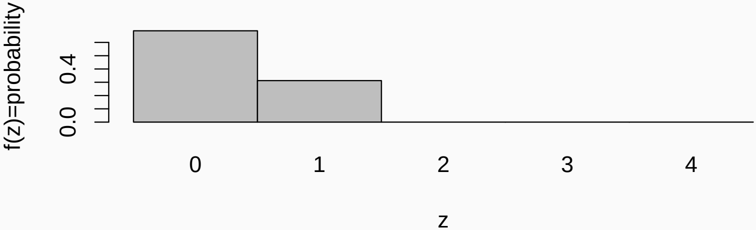
$$E(Y) = \sum_{y=0}^{4} y f(y)$$

Expected Value (Mean of a r.v.)

The random variable Z is a Bernoulli r.v.

$$Z = \begin{cases} 1 & \text{if more female children than male} \\ 0 & \text{otherwise} \end{cases}$$

z	0	1
$f(z)$	$11/16$	$5/16$



Make a guess. Approximate the average outcome for Z.

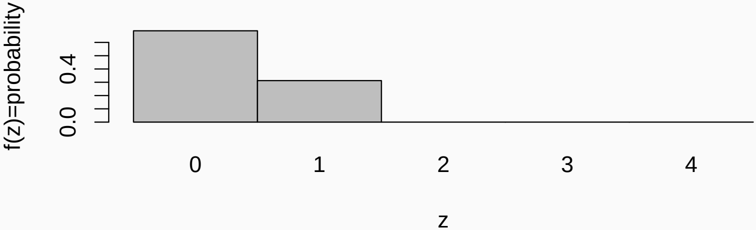
$E(Z) =$

Expected Value (Mean of a r.v.)

The random variable Z is a Bernoulli r.v.

$$Z = \begin{cases} 1 & \text{if more female children than male} \\ 0 & \text{otherwise} \end{cases}$$

z	0	1
$f(z)$	$11/16$	$5/16$



Make a guess. Approximate the average outcome for Z.

$$E(Z) = \sum_{z=0}^{1} zf(z) = (0)11/16 + (1)5/16 = 5/16.$$

Expected Value of a Bernoulli Random Variable

The random variable Z is a Bernoulli r.v.

$$Z = \begin{cases} 1 & \text{if more female children than male = "success"} \\ 0 & \text{otherwise = "failure"} \end{cases}$$

Probability mass function (PMF):

z	0	1
$f(z)$	$(1-p)$	p

where $p = 5/16$ = probability of a success.

$$\mu_Z = E(Z) = \text{weighted average of all possible outcomes } x$$

$$E(Z) = \sum_{\text{all } z} z\, P(Z = z) = \sum_{z=0,1} z\, P(Z = z)$$

$$= (0)(1-p) + (1)(p) = p = 5/16.$$

Expected Value of a Bernoulli Random Variable

The random variable Z is a Bernoulli r.v.

$$Z = \begin{cases} 1 & \text{if more female children than male} = \text{``success''} \\ 0 & \text{otherwise} = \text{``failure''} \end{cases}$$

Prob mass function (PMF): $\quad f(z) = p^z\,(1-p)^{1-z} \quad \text{for } z = 0, 1$

where $p = 5/16 =$ probability of a success.

$$\mu_Z = E(Z) = \text{weighted average of all possible outcomes } x$$

$$E(Z) = \sum_{\text{all } z} z\, f_Z(z) = \sum_{z=0}^{1} z\, p^z\,(1-p)^{1-z}$$

$$= (0)p^0(1-p)^{1-0} + (1)p^1(1-p)^{1-1} = p = 5/16.$$

The Binomial Setting

1. There is a fixed number of observations n.
2. The n observations are all independent.
3. Each observation falls into one of just two categories. For convenience, called “success” and “failure”
4. The probability of a success (p) is the same for each observation.

The Binomial Distribution

Let X = the count of successes in a Binomial setting

Then, the following statements are equivalent:

- X has a Binomial distribution with parameters n and p.
- X is a Binomial(n, p) random variable.
- $X \sim$ Binomial(n, p).
- X is the sum of n independent Bernoulli r.v. (***)
- The probability mass function (pmf) for random variable X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \text{for } x = 0, 1, ..., n$$

n = number of observations (sample size)

p = probability of success for any one observation

Expected Value of a Binomial Random Variable

Is X a Binomial(n, p) random variable?

Without studying, you plan to randomly guess each quiz question.

(1) $X =$ number of correct answers in a quiz with 10 questions and 5 choices per question (A, B, C, D, E).
(2) $X =$ number of correct answers in a quiz with 100 questions and 4 choices per question (A, B, C, D).
(3) $X =$ number of correct answers in a quiz with 50 questions and 4 choices per question (A, B, C, D).

In each case, how many correct answers do you expect to get?

Let $X = \#$ of "successes" in n independent Bernoulli (p) "trials"

Expected Value of a Binomial Random Variable

Mean of Binomial RV

If X is a Binomial(n, p) random variable, $E(X) = np$.

$$
\begin{aligned}
E(X) &= \sum_{\text{all } x} x \, f(x) \\
&= \sum_{x=0}^{n} x \binom{n}{x} p^x \, (1-p)^{n-x} \\
&= np
\end{aligned}
$$

We'll learn an easy way to prove this.