- Moment Generating Functions:

- Recall the Mean of a R.V. $g(x)$

$$E[g(x)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f(x)\, dx & \text{if } X \text{ is continuous} \\ \sum_x g(x) f(x) & \text{if } X \text{ is discrete} \end{cases}$$

$$Var(g(x)) = E\left[(g(x) - E[g(x)])\right]^2 = E[g(x)^2] - E[g(x)]^2$$

- Sometimes it is easy to derive $E[X^k]$ by its definition. Sometimes it's very hard.

- Def: MGF (Moment Generating Function):
Let X be a R.V. The MGF of X, denoted $M_X(t)$, is

$$M_X(t) = E[e^{tX}]$$

provided expectation exists in some neighborhood about 0.
i.e. $\exists\ h > 0$ s.t. Ee^{tX} exists for all $-h < t < h$

- "Moment" = fancy physics word for mean
1^{st} moment $= E[X^1] = E(X)$
2^{nd} moment $= E[X^2]$
3^{rd} moment $= E[X^3]$
$\vdots$
k^{th} moment $= E[X^k]$
$\vdots$

- Motivation:
① MGF is sometimes an easier way to find the means and variances of R.V.'s
② MGF can uniquely identify the distribution of an R.V. ⇒ useful for proofs.

- Thm: (Generating Moments)
If X has MGF $M_X(t)$, then
$$E[X^n] = M_X^{(n)}(0)$$
where
$$M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t) \Big|_{t=0}$$

n^{th} moment = n^{th} derivative of $M_X(t)$ evaluated @ $t=0$

- Generating 1st Moment:

① Find MGF of X:

$$M_X(t) = E[e^{tX}] = E(g(X)) \quad \text{where } g(X) = e^{tX}$$
$$= \sum_x g(x)f(x) \quad \text{if discrete}$$
$$\int_{-\infty}^{\infty} g(x) f(x)\, dx \quad \text{if continuous}$$

② Find derivative of $M_X(t)$ with respect to t

if $E[e^{tX}] < \infty$, then can switch order of expectation and differentiation

derivative of sum = sum of derivative

$$M'_X(t) = \frac{d}{dt} E[e^{tX}] = E\left[\frac{d}{dt} e^{tX}\right]$$
$$= E[Xe^{tX}]$$

③ Set $t = 0$

$$M'(0) = E[Xe^{0X}] = E[X] = 1^{st} \text{ moment}$$

- How do we generate $E[x^2]$?

① Calculate second derivative of $M_x(t)$ w.r.t. t

$$M''_x(t) = \frac{d}{dt} M_x(t) = \frac{d}{dt} E[x e^{tx}]$$

$$= E\left[\frac{d}{dt} x e^{tx}\right] \leftarrow \text{if } E[e^{tx}] < \infty \text{ and } E[x e^{tx}] < \infty$$

$$= E[x^2 e^{tx}]$$

② set $t = 0$

$$M''_x(0) = E[x^2 e^{0x}] = E[x^2]$$

- Since we know $E[x]$ and $E[x^2]$, we have

$$Var(x) = E[x^2] - E[x]^2$$

- MGF Example #1: $X \sim Bern(p)$

We know $E[x] = p$, $Var(x) = p(1-p)$
Let's prove this with the MGF method

① find $E[e^{xt}]$ s.t. $x \sim Bern(p)$

$$E[e^{xt}] = (1-p)e^{0t} + pe^{1t} = 1-p + pe^t$$

② Find derivative

$$M'_x(t) = \frac{d}{dt}(1-p+pe^t) = pe^t$$

③ set $t=0$,

$$M'_x(0) = pe^0 = p = E[x]$$

For $E[x^2]$:

① $M''_x(t) = \frac{d^2}{dt^2}(1-p+pe^t) = \frac{d}{dt} pe^t = pe^t$

② $M''_x(0) = pe^0 = p = E[x^2]$

$$\begin{aligned} Var(x) &= E[x^2] - E(x)^2 \\ &= p - p^2 \\ &= p(1-p) \end{aligned}$$

• <u>MGF Example #2:</u> $x \sim$ Binomial

We already know $E[x] = np$, $var(x) = np(1-p)$
Let's prove this using MGF's!

① $M_x(t) = E[e^{tx}] = \sum_x e^{tx} f(x)$

$$= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} e^{tx}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= [(1-p) + pe^t]^n$$

Binomial Theorem: For any real numbers w and y
$$\sum_{x=0}^{n} \binom{n}{x} w^x y^{n-x} = (w+y)^n$$

② $M'_x(t) = \frac{d}{dt}[(1-p) + pe^t]^n$

$$= n[(1-p) + pe^t]^{n-1} \frac{d}{dt}[(1-p) + pe^t] \leftarrow \text{chain rule}$$

$$= n[(1-p) + pe^t]^{n-1} pe^t$$

③ set $t=0$

$$M_x(0) = n\left[(1-p)+pe^0\right]^{n-1} pe^0$$
$$= n\left[1-\cancel{p}+\cancel{p}\right]^{n-1} p$$
$$= np$$

For variances, use MGF to find $E[x^2]$

① $M''_x(t) = \frac{d}{dt} M'_x(t)$

$$= \frac{d}{dt}\ n\left[(1-p)+pe^t\right]^{n-1} pe^t$$

$$= n(n-1)\left[(1-p)+pe^t\right]^{n-2} pe^t \overbrace{\frac{d}{dt}\left\{(1-p)+pe^t\right\}}^{=pe^t}$$
$$+ n\left[(1-p)+pe^t\right]^{n-1} pe^t$$

$$= n(n-1)\left[(1-p)+pe^t\right]^{n-2} p^2e^{2t}$$
$$+ n\left[(1-p)+pe^t\right]^{n-1} pe^t$$

② set $t=0$

$$E[x^2] = M''_x(0) = n(n-1)\left[(1-p)+pe^0\right]^{n-2} p^2 e^{2\cdot 0}$$
$$+ n\left[(1-p)+pe^0\right]^{n-1} pe^0$$

$$= n(n-1)\left[1-\cancel{p}+\cancel{p}\right]^{n-2} p^2$$
$$+ n\left[1-\cancel{p}+\cancel{p}\right]^{n-1} p$$

$$= n(n-1)p^2 + np = n^2p^2 - np^2 + np$$

③ $\mathrm{Var}(x) = E[x^2] - E[x]^2$

$$= \overbrace{\cancel{n^2p^2} - np^2 + np}\ \ \overbrace{- \cancel{n^2p^2}}$$

$$= np - np^2 = np(1-p)$$ //